Perturbation theory for the kink of the sine-Gordon equation

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A singular perturbation theory is developed to investigate the kink propagating in systems governed by the sine-Gordon equation with perturbations. The outstanding characteristic of the present theory lies in that the dynamic equation and the dispersive wave as well as the ''translation mode'' are consistently determined in a natural manner, involving no sophisticated derivations pertaining to the inverse scattering transform. A distinct and strict linearization for the subject is introduced. Some notable cases are reformulated by the theory.

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The study of solitons under the influence of perturbations is a subject of considerable significance for both application and theoretical researches. From the point of view of the modern perturbation theory, the regular expansion is always invalid even for linear systems suffering from weakly nonlinear perturbations [1], which certainly hold true for solitons affected by perturbations. Among a number of techniques to handle this kind of problems, the most popular and effective one utilized to determine an asymptotic expansion that is uniformly valid is the technique of multiple scales [1]. It was actually employed in the development of perturbation theories for the KDV equation [2] and the nonlinear Schrödinger equation [3] in an attempt to prevent the occurrence of secular terms. Now, it is universally acknowledged that, at the presence of perturbations, solitons not only modify their shape by a correction of a linear dispersive wave but also undergo a slow change of their parameters [4]. These are two basic features of the soliton's perturbation problems. We realized that a perturbation theory for a soliton should depict these in a natural manner, which will provide a more direct insight into fundamental aspects of such problems. The so-called ''two timing'' technique, or its archetype of the more general method of multiple scales, should be a canonical way to characterize these problems in the framework of direct perturbation approaches.

In the study of nonlinear wave phenomena the sine-Gordon equation frequently emerges from a vast range of physical applications [4,5]. As an important example, the model of long Josephson transmission line, which recently received renewed interest due to the appearance of high-temperature superconductors [6], provides us an applicable problem to investigate the kink's dynamics under the action of external force and some dissipations. This topic received much attention in previous studies as well [7,8]. It is well known that the sine-Gordon equation is an integrable one and possesses a number of remarkable mathematical properties. The most celebrated inverse scattering transform can be used to formulate the kink solution and its perturbation theory [9]. But this elaborate mathematical feat seems a little abstruse for most physical researchers, and it is difficult to derive an explicit expression for the dispersive wave from an associated linear integral equation. A two-stage scheme was thus proposed by McLaughlin and Scott [7]. They first computed slow modulations of velocity and initial position of the kink, and then calculated the first-order correction through a constructed radiative Green's function. They stated that their method is partially based upon the inverse scattering transform, which alludes to the sophisticated construction of the representation. Fogel *et al.* [10] put forward a direct perturbation theory for the subject. They paid attention mainly to the kink with low velocity in comparison with its limiting value of unity, based on their expectation that perturbations have little influence on the kink with large velocity. This theory is founded on the basis of eigenfunctions of a Schrödinger operator being of a simple form. Some of their results [10] were discussed by others [11,12]. Moreover, in this scheme, Flesch and Trullinger [13] investigated the static kink. Analytical forms for the Green's function are derived and expressed in terms of ''modified'' Lommei functions. In addition to above formal theories, there is still a more popular and simpler approximate scheme designated to determine moving equations for the kink [12,14] by making use of modified conservation laws. An apparent disadvantage of this scheme is generally remarked to be that it is useless for the understanding of linear elementary excitations in the system.

From the above introduction, a plain and general theory is needed for this important subject. In the present paper, a direct perturbation approach to investigate the motion of kink in the systems described by the sine-Gordon equation with perturbations will be developed. In this scheme, we employ the derivative expansion method to linearize the perturbed sine-Gordon equation in the coordinate frame attached to the moving kink. In order to eliminate potential secular terms in the solution, parameters of the kink are first assumed to be dependent on slow time scales. Although this distinct process of linearization for the perturbed sine-Gordon equation involves somewhat complicated calculations, its idea is plain and strict. After the linearization, we take the Laplace transform to reduce the linearized equation to an ordinary differential equation that is, by virtue of a further function transform, converted into a form appropriate for solution by the method of eigenexpansion. Naturally, an eigenvalue problem that is not self-adjoint is extracted from our derivation, and its eigenfunctions are used to construct a complete basis underlying our solution. The solution turns out to contain two types of secular terms that are directly proportional to t and t^2, respectively. Imposition of secular conditions results in two equations governing the slow variation of the parameters in time. The final solution for first-order correction consists of two branches of dispersive wave traveling in opposite directions and a localized state usually referred to as the ‘‘translation mode.’’

We start with the perturbed sine-Gordon in the form

$$U_{tt}-U_{xx}+\sin U=\varepsilon P[U], \tag{1}$$

where ε is a small positive parameter, and $P[U]$ is a function of U and its derivatives with respect to time and space. If perturbations are absent, i.e., by setting ε to zero, Eq. (1) admits a kink solution given by

$$U(t,x)=4\ \arctan\ e^{m(x-vt-\chi')}, \tag{2}$$

where v and χ' signify the kink's velocity and initial position, respectively, and $m=1/\sqrt{1-v^2}$. Now, we consider Eq. (1) with an initial state $U(0,x)=4\ \arctan e^{m(x-\chi')}$. In this case, the initial profile cannot travel as that described by Eq. (2), but it is reasonable to suppose that it is a slowly varying kink shape plus a small correction. Thus, we first introduce a series of slow time scales $t_n=\varepsilon^n t$ and then write the solution of Eq. (1) as

$$U(t,z,\{t_n\})=U^{(0)}(z)+\varepsilon u(t,z,\{t_n\})+\text{higher-order terms}, \tag{3}$$

where $U^{(0)}(z)=4\ \arctan e^z$ figures a moving kink, $z=m[x-\varepsilon^{-1}\chi(\{t_n\})-\chi'(\{t_n\})]$ is the coordinate variable in the frame co-moving with the kink and the higher-order terms of ε are neglected in our subsequent calculations. Since these time scales introduced above will be treated as independent variables, the derivative with respect to time should be replaced by $\partial_t=\partial_t+\varepsilon\partial_{t_1}+\cdots$, which is the so-called derivative expansions [1]. If we further select t, z, t_1 as independent variables and just consider up to the first order of ε, the second-order temporal and spatial derivatives in Eq. (1) must be replaced by

$$\begin{aligned}\frac{\partial^2}{\partial t^2}=&\frac{\partial^2}{\partial t^2}-2ma\frac{\partial^2}{\partial t\partial z}+m^2a^2\frac{\partial^2}{\partial z^2}-\varepsilon 2am_{t_1}\left(\frac{\partial}{\partial z}+z\frac{\partial^2}{\partial z^2}\right)\\&+\varepsilon 2m^2a\chi'_{t_1}\frac{\partial^2}{\partial z^2}-\varepsilon ma_{t_1}\frac{\partial}{\partial z}+\varepsilon\frac{2m_{t_1}}{m}z\frac{\partial^2}{\partial t\partial z}\\&-\varepsilon 2m\chi'_{t_1}\frac{\partial^2}{\partial t\partial z}-\varepsilon 2ma\frac{\partial^2}{\partial t_1\partial z}+\varepsilon 2\frac{\partial^2}{\partial t\partial t_1}\end{aligned} \tag{4}$$

and

$$\frac{\partial^2}{\partial x^2}=m^2\frac{\partial^2}{\partial z^2}, \tag{5}$$

where $a=\chi_{t_1}$ is defined for later convenience. Substitution of Eqs. (3)–(5) into Eq. (1) yields a sequence of equations for each power of ε. The zeroth- and first-order ones read

$$m^2(a^2-1)U_{zz}^{(0)}+\sin U^{(0)}=0, \tag{6}$$

$$u_{tt}-2mau_{tz}+m^2(a^2-1)u_{zz}+(\cos U^{(0)})u=M(z). \tag{7}$$

We would like to indicate that we are now in the frame comoving with the kink. Obviously, the zeroth-order Eq. (6) requires that $m=1/\sqrt{(1-a^2)}$. In consequence, the first-order Eq. (7) becomes

$$u_{tt}-2mau_{tz}-[u_{zz}+(2\ \text{sech}^2\ z-1)u]=M(z) \tag{8}$$

with

$$\begin{aligned}M(z)=&P(z)+2ma_{t_1}\psi_0(z)+4m^3a^2a_{t_1}\overline{\psi_0}(z)\\&+4m^2a\chi'_{t_1}\overline{\psi_1}(z),\end{aligned} \tag{9}$$

where $\psi_0(z)=\text{sech}\,z$, $\overline{\psi_0}(z)=\text{sech}\,z(1-z\tanh z)$, and $\overline{\psi_1}(z)=\text{sech}\,z\tanh z$. Thus the distinct linearization for the sine-Gordon equation has been completed. Apart from the neglect of high-order terms of ε, no approximation is involved. The reason we carry out such a somewhat complicated process is to preliminary provide extra freedoms to remove secular terms that will appear in the solution of Eq. (8). Now, we proceed with the solution by use of Laplace transform to Eq. (8), which gives

$$s^2\tilde{u}-2mas\tilde{u}_z-[\tilde{u}_{zz}+(2\ \text{sech}^2\ z-1)\tilde{u}]=s^{-1}M(z). \tag{10}$$

Equation (10) is hard to handle directly; hence we devise two function transforms to reduce it into forms appropriate for solution by the method of eigenfunction decomposition. By letting $\tilde{u}=ve^{-m(a+1)sz}$ and $\tilde{u}=v'e^{-m(a-1)sz}$, respectively, and inserting them into Eq. (10), we get

$$2msv_z-\hat{L}v=s^{-1}M(z)e^{m(a+1)sz}, \tag{11}$$

$$2msv'_z+\hat{L}v'=-s^{-1}M(z)e^{m(a-1)sz}, \tag{12}$$

where $\hat{L}$ is an ordinary differential operator defined by $\hat{L}=d^2/dz^2+(2\ \text{sech}^2\ z-1)$. Equations (11) and (12) are inhomogeneous ordinary differential equations in nature. As usual, we must first consider its homogeneous counterpart, which will result in the following eigenvalue problem:

$$\hat{L}\psi=\lambda\psi_z. \tag{13}$$

The above eigenvalue problem is apparently not self-adjoint, its eigenstates consist of a continuous spectrum $\psi(z,k)$ with the eigenvalue $\lambda=i(k^2+1)/k$ and a discrete state $\psi_0(z)$ with the eigenvalue $\lambda=0$. Under the definition of inner product in Hilbert space, these states are orthogonal, but not complete; an extra orthogonal discrete state $\psi_1(z)=z\,\text{sech}\,z$ must be appended to the eigenstates to complete the closure relation. This phenomenon comes out to be popular in the direct soliton perturbation theories [15]. We recall that, in the previous theories, one usually derived an incomplete eigenstate from a partial differential operator by virtue of a smart relation with the inverse scattering transform and then completed them, which is really cumbersome. In the present scheme, a set of complete basis $\{\psi(z,k),\psi_0(z),\psi_1(z)\}$ is easily constructed from eigenstates of Eq. (13). The explicit expression for the continuous spectrum is given by

$$\psi(z,k)=\frac{1}{\sqrt{2\pi}(k^2+1)}(1-k^2-2ik\tanh z)e^{ikz}. \quad (14)$$

The corresponding adjoint states of the above set consist of $\{\overline{\psi}(z,k),\overline{\psi}_0(z),\overline{\psi}_1(z)\}$. The continuous spectrum of adjoint states is calculated by the relation $\overline{\psi}(z,k)=\psi_z(z,k)/ik$, which reads

$$\overline{\psi}(z,k)=\frac{1}{\sqrt{2\pi}(k^2+1)}(1-k^2-2ik\tanh z-2\,\mathrm{sech}^2 z)e^{ikz}. \quad (15)$$

There exists a relation between the noneigenstate and its adjoint, namely, $\hat{L}\psi_1(z)=-2\overline{\psi}_1(z)$. These relations will be used in the later derivation. Based on this set we can decompose $v(s,z)$ in Eq. (11) as

$$v(s,z)=\int_{-\infty}^{+\infty}dk\tilde{v}(s,k)\psi(z,k)+\tilde{v}_0(s)\psi_0(z)+\tilde{v}_1(s)\psi_1(z). \quad (16)$$

By virtue of expansion Eq. (16), we can derive $v(s,z)$ from Eq. (11) without difficulty. Recalling that $\tilde{u}=ve^{-m(a+1)sz}$, and taking the inverse Laplace transform for $\tilde{u}(s,z)$, we get

$$\begin{aligned}u(t,z)=&\int_{-\infty}^{+\infty}dk\frac{1}{k^2+1}\int_{-\infty}^{+\infty}dz'[1-e^{i[(k^2+1)/2mk]\beta^+}]\\&\times M(z')\psi^*(z',k)\psi(z,k)\\&+\frac{1}{2m}\int_{-\infty}^{+\infty}dz'\beta^+M(z')\psi_1(z')\psi_0(z)\\&-\frac{1}{4m^2}\int_{-\infty}^{+\infty}dz'(\beta^+)^2M(z')\psi_0(z')\psi_0(z), \quad (17)\end{aligned}$$

where $\psi^*(z,k)$ represents the complex conjugate of $\psi(z,k)$ and $\beta^+=[t+m(a+1)(z'-z)]$ is defined. It is apparent that secular terms directly proportional to t and t^2 occur in the second and third terms in the above solution. Removing them, we must impose

$$\int_{-\infty}^{+\infty}M(z')\psi_0(z')dz'=0 \quad (18)$$

and

$$\int_{-\infty}^{+\infty}M(z')\psi_1(z')dz'=0, \quad (19)$$

which are customarily referred to as secular conditions. Hence we get the final solution

$$\begin{aligned}u(t,z)=&\int_{-\infty}^{+\infty}dk\frac{1}{k^2+1}\int_{-\infty}^{+\infty}dz'[1-e^{i[(k^2+1)/2mk]\beta^+}]\\&\times M(z')\psi^*(z',k)\psi(z,k)\\&+\frac{1}{4m^2}\int_{-\infty}^{+\infty}M(z')z'^2\psi_0(z')dz'\psi_0(z). \quad (20)\end{aligned}$$

The first term in the solution corresponds to the dispersive wave traveling along the positive z direction; the second term is the "translation mode." Following absolutely the same procedure, we can acquire another solution via Eq. (12),

$$\begin{aligned}u(t,z)=&\int_{-\infty}^{+\infty}dk\frac{1}{k^2+1}\int_{-\infty}^{+\infty}dz'[1-e^{-i[(k^2+1)/2mk]\beta^-}]\\&\times M(z')\psi^*(z',k)\psi(z,k)\\&+\frac{1}{4m^2}\int_{-\infty}^{+\infty}M(z')z'^2\psi_0(z')dz'\psi_0(z), \quad (21)\end{aligned}$$

where $\beta^-=[t+m(a-1)(z'-z)]$. The secular conditions and the "translation mode" are just the same as preceding results, which is very reasonable, but the dispersive wave travels in the opposite direction. Inserting Eq. (9) into Eqs. (18) and (19), we have

$$a_{t_1}=-\frac{1}{4m^3}\int_{-\infty}^{+\infty}P(z)\mathrm{sech}\,z\,dz, \quad (22)$$

$$\chi'_{t_1}=-\frac{1}{4m^2a}\int_{-\infty}^{+\infty}P(z)z\,\mathrm{sech}\,z\,dz, \quad (23)$$

which govern the slow variation of the kink's velocity and initial position in time.

To further display the intrinsic aspect of this scheme, we give our different viewpoint of a notable example of the kink under the action of a small constant external force corresponding to the dc bias current in the long Josephson junction and a dissipative loss resulting from tunneling of normal electrons across the barrier, namely, $P[U^{(0)}]=\gamma-\eta U_t^{(0)}$. In the case of $P=\gamma$, since the right-hand side of Eq. (11) approaches infinity as $z\to+\infty$ and so does Eq. (12) as $z\to-\infty$, some modifications must be made to our theory. This problem can be settled by dividing the solution of Eq. (10) into two parts,

$$s^2\tilde{u}^{(1)}-2mas\tilde{u}_z^{(1)}-\hat{L}\tilde{u}^{(1)}=s^{-1}\theta(-z)M(z), \quad (24)$$

$$s^2\tilde{u}^{(2)}-2mas\tilde{u}_z^{(2)}-\hat{L}\tilde{u}^{(2)}=s^{-1}\theta(z)M(z), \quad (25)$$

in which $\theta(z)$ is the Heaviside function. It can be easily verified that $\tilde{u}=\tilde{u}^{(1)}+\tilde{u}^{(2)}$ is our desired solution. Solving Eq. (24) as that for Eq. (11) and Eq. (25) as that for Eq. (12), incorporating these two solutions, we get

$$u(t,z)=\int_{-\infty}^{+\infty}dk\,\frac{1}{k^2+1}\int_{-\infty}^{+\infty}dz'[1-e^{i[(k^2+1)/2mk]\beta^+}]$$
$$\times\theta(-z')M(z')\psi^*(z',k)\psi(z,k)$$
$$+\int_{-\infty}^{+\infty}dk\,\frac{1}{k^2+1}\int_{-\infty}^{+\infty}dz'[1-e^{-i[(k^2+1)/2mk]\beta^-}]$$
$$\times\theta(z')M(z')\psi^*(z',k)\psi(z,k)$$
$$+\frac{1}{4m^2}\int_{-\infty}^{+\infty}M(z')z'^2\psi_0(z')dz'\,\psi_0(z). \qquad (26)$$

In the modification, relations for variation of the velocity and initial position remain unchanged. Thus, $a_{t_1}=-\gamma(1/4m^3)\int_{-\infty}^{+\infty}\operatorname{sech} z\,dz$. Recalling that $\chi_{t_1}=a$ and $t_1=\varepsilon t$, by integration, we derive $\chi(\varepsilon,t)=[m_0-\sqrt{(m_0a_0-\varepsilon\alpha t)^2+1})]/\alpha$, where $m_0=1/\sqrt{(1-a_0^2)}$, $\alpha=\pi\gamma/4$, and a_0 is the initial velocity of the kink.

Now, let us proceed to the next perturbation $P[U^{(0)}(z)]=-\eta U_t^{(0)}=2ma\eta\operatorname{sech} z$. From Eq. (22) we have $a_{t_1}=-a\eta/m^2=-a(1-a^2)\eta$. Integrating this equation from 0 to t_1 yields $a\sqrt{(1-a)/(1+a)}=a_0\sqrt{(1-a_0)/(1+a_0)}e^{-\varepsilon\eta t}$. Considering a small value of the initial velocity, we find that the initial velocity exponentially decreases, viz., $a(\varepsilon,t)=a_0e^{-\varepsilon\eta t}$, and the equation of motion is given by $\chi(\varepsilon,t)=a_0(1-e^{-\varepsilon\eta t})/\eta$.

We have demonstrated a theory for the study of the sine-Gordon equation under perturbations. The slow variation of parameters in time and the first-order correction consisting of two branches of dispersive wave traveling in opposite directions and a localized state are determined in a consistent manner. It can be seen that no advanced and sophisticated mathematical techniques are necessary for the theory. It is constructed in a normal frame in the modern perturbation theory and can be viewed as a successful implementation of the powerful technique and idea of the method of multiple scales in the perturbation theory of solitons. Hence we have reason to believe that it is a more reasonable and natural way in comparison with previous theories. Here, we should note that the theory by MacLaughlin and Scott [7] is canonical. Although it is principally devised for the multikink problem, it is actually of practical sense for the single kink. The basic idea of their theory is very plain, but they came up against a partial differential operator in matrix form that is hard to manage directly and necessary for sophisticated mathematical tools. It is fortunate enough that the representation for the Green's function is found to be constructed by virtue of a ''squared eigenfunction'' from the inverse scattering method or by the use of Bäcklund transformations, which is really an exhibition of remarkable properties of the sine-Gordon equation. The theory by Fogel *et al.* [10] is in the framework of regular expansion, and is usually mentioned as the ''collective coordinate method.'' Their result, that the kink behaves as a Newtonian particle in their theory, caused some controversy by a number of authors [11,12]. In fact, this one is included in our theory by introducing the same approximation, i.e., a is small, whereas we hold extra freedom to handle the secular term.

In conclusion, nonlinear evolution equations underlying solitons turn out to share a series of special properties, which is certainly true for its perturbed counterpart. Hereby, it is expected that some theoretical structure more intrinsic should be discovered and illustrated at the perturbation level. In this sense, our theory reveals a facet of perturbation theory of solitons.

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