Phase synchronization and its cluster feature in two-dimensional coupled map lattices

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Due to a diffusive nearest-neighbor coupling, phase-synchronized states can emerge in two-dimensional chaotic coupled map lattices. By defining a direction phase (like a spin with up or down direction) as the direction of two sequential iterations of the logistic map, we find several novel kinds of phase synchronization which correspond to four different regions in a phase diagram. For the phase with partial phase synchronization, as the coupling strength ϵ increases to a critical threshold ϵ_c, a percolationlike transition is found in the cluster feature of the direction phases relating to the pattern formation. In addition, a scaling of the percolation probability $\rho \sim (\epsilon-\epsilon_c)^\beta$ with $\beta=2.1$ is obtained. The spatial and time correlation functions of the phase clusters are also discussed.

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I. INTRODUCTION

It is well known that an antiferromagnet experiences a phase transition to a ferromagnet because of the interaction between the spins when the temperature is lowered to a critical one. In the ferromagnetic state, the magnetization is not equal to zero, implying a net "up" order of the spins. The appearance of such an order relates to the symmetry breaking in the system. At the same time, there are some scaling exponents which well characterize such a transition [1]. Recently, phase-transition or phase-transition-like phenomena have received much attention in chaotic dynamic systems, such as in the coupled map lattices (CMLs) [2–13] and some other systems (see Ref. [6] and the references therein). Especially, the coupled map lattices have become a general model for testing physical intuitions and concepts in the study of spatiotemporal chaotic systems. By separating the values of iterations of each site in the systems into different regions or its signs into positive or negative, the systems are argued to be similar to the lattice model of Ising spins, with "up" or "down" spin. Then, some clusters with up-spin or down-spin are obtained, and the collective dynamic behavior of such spins and the dynamic scaling of some of the physical properties, e.g., the magnetization of the systems, are well studied. However, the phase of systems is obviously of the map dependence since different maps may show different phases [2–19].

Generally, in a dynamic system, a chaotic state means that the system follows a chaotic trajectory and behaves with randomlike features. For a trajectory of a continuous dynamic system, the phase can be well defined by its tangent direction in the phase space. Stimulated by a similar idea, one of the authors (W.W.) of the present work and his co-worker found a new phase-ordering in a single logistic map and the corresponding pattern of a spatially phase order state in the CML by defining a direction phase for a discrete dynamical map [13]. Such a direction phase is defined by the difference between two sequential values of iterations of a map. There is a transition for the phase order via a symmetry breaking in the chaotic map which results in a state of nonzero net direction phase, or a transition from an ordered arrangement of the direction phases to a disordered one.

In this paper, we report a study on the synchronization of the direction phase for two-dimensional coupled chaotic map lattices and their cluster feature. From our simulations we obtain a phase diagram which shows four different synchronization patterns of the direction phase. We find a percolationlike transition in the region with the partial synchronization pattern of the direction phases. This transition shows a different scaling behavior from the ordinary geometrical percolation. The spatial and time correlation functions are also found. These functions depend closely on the coupling strength.

This paper is organized as follows. In Sec. II, we describe our model of two-dimensional coupled map lattices with diffusive nearest-neighbor coupling. In Sec. III, we obtain a phase diagram, which shows four different phase synchronization patterns, by studying the phase synchronization behaviors for different coupling strengths ϵ and different parameters μ. In Sec. IV, we investigate the percolation feature of the phase clusters for the partial phase synchronization. Finally, in Sec. V we give a summary.

II. THE MODEL

Let us consider a two-dimensional lattice consisting of $L\times L$ sites with nearest-neighbor coupling [13],

$$\begin{aligned} x_{n+1}(i,j) = {} & f(x_n(i,j)) + \epsilon[f(x_n(i+1,j)) \\ & + f(x_n(i-1,j)) + f(x_n(i,j+1)) + f(x_n(i,j-1)) \\ & - 4f(x_n(i,j))]/4, \end{aligned} \tag{1}$$

where ϵ is the coupling strength and the function $f(x)$ is chosen as

$$f(x) = \mu x(1-x), \tag{2}$$

where $\mu \in [1,4]$. The lattice consists of $L\times L$ logistic maps, and all the maps are coupled together. The coupling is taken as the diffusive interaction, and its value of the coupling

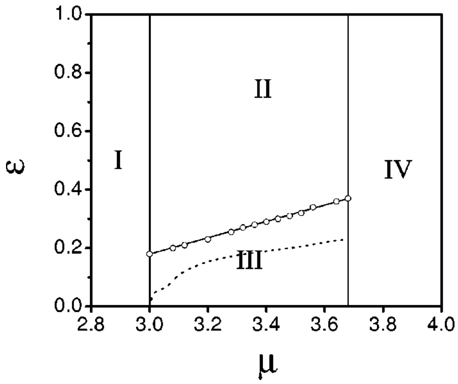

FIG. 1. Phase diagram: (I) stationary state, (II) full phase synchronization, (III) partial phase synchronization, (IV) complex phase synchronization. The dashed line corresponds to the percolation threshold of the phase clusters. The dividing line between phase II and phase III is given by the relation $\epsilon_0(\mu) \approx 0.28\mu - 0.67$.

strength ϵ is in the region of $[0,1]$. A periodic boundary condition is used, i.e., $x(0,j)=x(L,j), x(i,0)=x(i,L), x(L+1,j)=x(1,j), x(i,L+1)=x(i,1)$.

Like a single logistic map, the values of iterations in two-dimensional coupled map lattices $x_n(i,j)$ also display the direction of two sequential iterations. As with increasing the coupling strength ϵ, more and more sites have a similar oscillatory behavior. For these sites, the values of $x_n(i,j)$ always change their direction alternatively, i.e., first up-direction and then down-direction. We call the sites with the same direction phase at the same time phase synchronization sites. In order to quantitatively characterize the phase synchronization behavior in the two-dimensional coupled map lattice systems, the direction phase of the site (i,j) at some time n is defined as [13]

$$S_n(i,j)=\begin{cases} +1 & \text{if} \quad x_{n+1}-x_n>0, \\ -1 & \text{if} \quad x_{n+1}-x_n\leqslant 0. \end{cases} \tag{3}$$

Here $S_n(i,j)=1$ means the up-phase $S_\uparrow$ and $S_n(i,j)=-1$ means the down-phase $S_\downarrow$. The sites with the same $S_n(i,j)$ denote phase synchronization and the phase clusters are composed of those sites with the same direction phase.

III. PHASE DIAGRAM

Starting from random initial conditions, we make the iterations for Eq. (1) in the parameter region with $\epsilon \in [0,1]$ and $\mu \in [1,4]$. We find four different patterns of the phase synchronization shown in the phase diagram of Fig. 1.

(i) Phase I. When $\mu<3.0$, after many iterations, the values of $x_n(i,j)$ for each site reach a fixed point, i.e., $x_{n+1}=x_n$. So the direction phase $S_n(i,j)$ is always -1 according to the definition in Eq. (2). The phase series of each site is $\downarrow\downarrow\downarrow\downarrow\downarrow\cdots$ for $n=1,2,3,\ldots$ and so on. It is a stationary state.

(ii) Phase II. In the region of $3.0<\mu<\mu_0$ ($\mu_0=3.68$), as the coupling strength ϵ increases, more and more sites have a similar oscillation. When ϵ reaches a critical value ϵ_0, all sites of the lattice have the same direction phase at the same time. The phase series of each site is $\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\cdots$ for $n=1,2,3,\ldots$, i.e., all sites of the lattice have the up-phase at time n and then the down-phase at time $n+1$. In this case, the phase cluster covers the whole lattice. We call this situation full phase synchronization. The critical value ϵ_0 is found to have a dependence on the parameter μ, i.e., $\epsilon_0=\epsilon_0(\mu)$. Our simulations give the relationship between ϵ_0 and μ as

$$\epsilon_0(\mu) \approx 0.28\mu - 0.67, \tag{4}$$

which gives a line of demarcation between phase II and phase III.

(iii) Phase III. When the parameter μ is in the region of $3.0<\mu<\mu_0$, but the coupling strength ϵ is below the critical value ϵ_0, some sites of the lattice have the up-phase and others have the down-phase at a certain time n. The phase series is $\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\cdots$ for some sites and $\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\cdots$ for other sites when $n=1,2,3,\ldots$, i.e., the sites with up-(down-) phase at time n will translate to down- (up-) phase at time $n+1$. In this phase, there exist phase clusters composed of the sites with the same direction phase. We refer to phase III as partial phase synchronization. The dashed line in this phase corresponds to a percolation threshold of the phase clusters, which will be discussed in the next section.

(iv) Phase IV. When $\mu>\mu_0$, the phase synchronization become complex. The direction phase does not always change alternatively with time like in phase II and III. It is possible for two consecutive iterations to have the same direction phase, such that, e.g., the direction phase series for $n=1,2,3,\ldots$ will be $\uparrow\uparrow\downarrow\uparrow\downarrow\downarrow\cdots$. We call this phase complex phase synchronization. In this case, there exist net orientations (up or down direction) for the time average of the clusters.

IV. PHASE CLUSTER FEATURE

Figure 2 shows the snapshots of the phase clusters in the case of $\mu=3.6$ for three different coupling strengths. We can see that when $\epsilon=0.15$ and $\epsilon=0.25$ [see Figs. 2(a) and 2(b)], which belongs to the region of phase III, some sites have up-phase (black dots) and others have down-phase (empty area). As the coupling ϵ increases, the direction phase changes from partial phase synchronization to full phase synchronization. The cluster size, which is defined as the number of sites in the same cluster, increases as the coupling ϵ increases, and when $\epsilon>\epsilon_0$ [$\epsilon_0=0.34$ in the case of $\mu=3.6$ from Eq. (3)] all sites have the same direction phase [see Fig. 2(c) in the case of $\epsilon=0.5$].

It should be noticed that, in phase III, as ϵ increases from 0 to ϵ_0, there exists a percolation transition for the connectivity of the clusters. From Fig. 2(a), we can see that when $\epsilon=0.15$, there are many small clusters dispersed on the whole lattice. When $\epsilon=0.25$, the sizes of the clusters become large and there is an infinite cluster throughout the whole network. This shows clearly the percolation behavior [20]. Generally, in the region of $[0,\epsilon_0]$, there exists a percolation threshold ϵ_c where an infinite cluster is formed. Thus

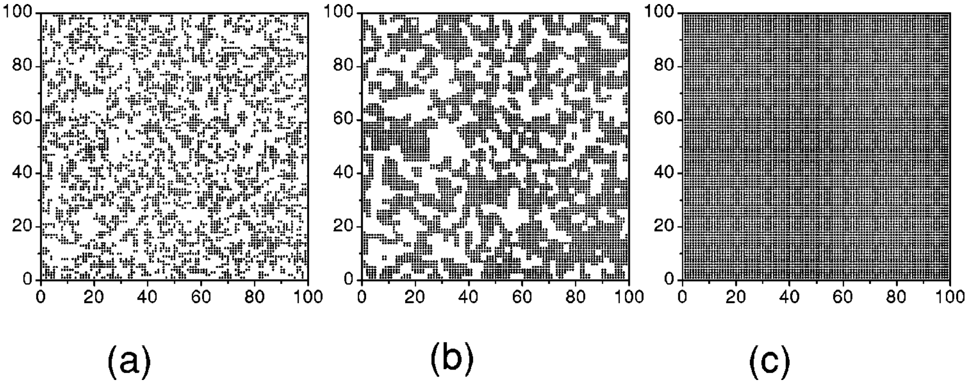

FIG. 2. The snapshots of the phase clusters with (a) $\epsilon = 0.15$, (b) $\epsilon = 0.25$, and (c) $\epsilon = 0.5$ in the case of $\mu = 3.6$. The black dots correspond to sites having up-phase and the empty area corresponds to sites having down-phase.

ϵ_c may indicate a phase transition such that for ϵ above ϵ_c one percolating network exists and for ϵ below ϵ_c no percolating network exists. The percolation threshold for a different parameter μ in phase III is shown with a dashed line in Fig. 1.

In the percolation model [21], the percolation probability ρ is chosen as the order parameter. It is defined as the probability of a site belonging to an infinite cluster. We perform computer simulations to calculate the percolation probability ρ for the systems with different parameters μ. Figure 3 shows the dependence of the percolation probability ρ on the coupling strength ϵ for three different lattice sizes in the case of $\mu = 3.6$. For each lattice size, there exists a critical value ϵ_c. The percolation probability ρ vanishes below ϵ_c and is nonzero above ϵ_c, and then reaches the value of unity very fast. The percolation threshold ϵ_c is found depending on the lattice size, i.e., $\epsilon_c = \epsilon_c(L)$. In the limit of infinite lattice, $L \to \infty$, the percolation threshold ϵ_c can be obtained from the extrapolation (see the inset in the right-bottom of Fig. 3), and in the case of $\mu = 3.6$ we find that ϵ_c is about 0.22. Similar to the ordinary geometrical percolation, we study the scaling between ρ and $\epsilon - \epsilon_c$ in a region near ϵ_c. We can define a critical exponent β by postulating $\rho \sim (\epsilon - \epsilon_c)^\beta$ for ϵ slightly above ϵ_c. In the left-top inset of Fig. 3 we give a log-log plot of the percolation probability ρ against $\epsilon - \epsilon_c$ for three different lattice sizes L. The straight lines through the data points indicate a power-law dependence of ρ on $\epsilon - \epsilon_c$. From the slopes of the lines the exponent β is found as $\beta = 2.1$ for all three different lattice sizes. Comparing with the value of $\beta = 0.14$ for the two-dimensional site percolation model, we find that the phase cluster percolation in a two-dimensional coupled map lattice may belong to a different universality from the ordinary geometrical percolation. This may be due to different physical reasons, such as the strong correlation between the dynamics of the neighboring sites.

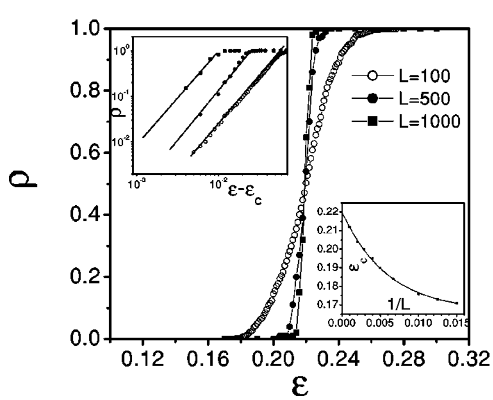

FIG. 3. ρ vs ϵ for three different lattice sizes when $\mu = 3.6$. The inset in the right-bottom gives the percolation threshold $\epsilon_c(L)$ for various lattice sizes L, and the value of ϵ_c in the limit of $L \to \infty$ can be obtained from the extrapolation. The inset in the left-top gives the scaling $\rho \sim (\epsilon - \epsilon_c)^\beta$ with $\beta = 2.1$.

Next, we study the correlation among the phase clusters. The correlation length ξ is defined as the average cluster size except for the infinite cluster. Figure 4 shows the correlation length ξ versus the coupling strength ϵ for several different lattice sizes. We can see that when $\epsilon < \epsilon_c$, where the infinite cluster is not formed yet, the correlation length ξ increases as ϵ increases. When $\epsilon > \epsilon_c$, there exists an infinite cluster. As ϵ increases, the correlation length ξ decreases due to more and more clusters merging to the infinite cluster, and when ϵ reaches ϵ_0 (here $\epsilon_0 = 0.34$ for $\mu = 3.6$), all lattice sites belong to the infinite cluster and the correlation length ξ vanishes. It is obvious that ξ reaches its maximum when $\epsilon = \epsilon_c$, and in the limit of $L \to \infty$, ξ becomes infinite near the percolation threshold ϵ_c.

The spatial and time correlations are defined with the following functions:

$$C(r) = \frac{\sum_{r_0} S_n(r_0) S_n(r_0 + r)}{\sum_{r_0} |S_n(r_0) S_n(r_0 + r)|}, \tag{5}$$

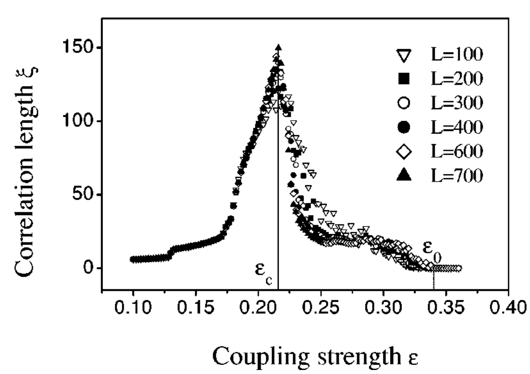

FIG. 4. The correlation length ξ vs ϵ for various lattice sizes L near the percolation threshold ϵ_c in the case of $\mu = 3.6$.

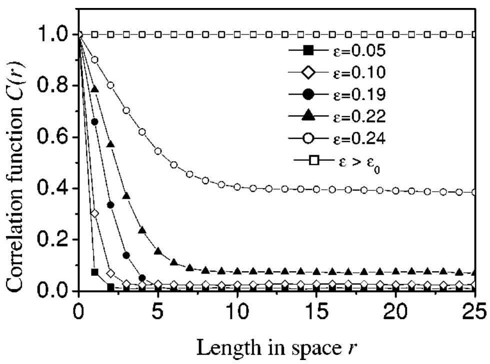

FIG. 5. Relationship between the spatial correlation function and the length in space in the case of $\mu=3.4, L=500$ for different coupling strengths $\epsilon=0.05,0.10,0.19,0.22,0.24,>\epsilon_0$. The critical value ϵ_0 is 0.28.

$$C(\Delta t)=\frac{\sum_n S_n(i,j)S_{n+\Delta t}(i,j)}{\sum_n |S_n(i,j)S_{n+\Delta t}(i,j)|}, \tag{6}$$

where $C(r)$ is the spatial correlation function for the distribution of phase synchronization in space, and $C(\Delta t)$ is the time correlation function for the evolution of phase synchronization in time. Here r and Δt are the spatial and time lags, respectively. Figure 5 shows the relationship between the spatial correlation function $C(r)$ and the spatial length r in the case of $\mu=3.4$ for different coupling strengths ϵ. Here, the lattice size is chosen as 500×500 sites. We can see that when $\epsilon \leqslant \epsilon_c$ ($\epsilon_c=0.19$ for $\mu=3.4$), the correlation function $C(r)$ is almost zero except for the spatial lags being within five sites. It is a short-distance correlation. As ϵ increases from ϵ_c, the spatial correlation becomes nonzero and shows stronger and stronger long-distance correlativity. When ϵ is above ϵ_0 ($\epsilon_0=0.29$ for $\mu=3.4$), all lattice sites belong to a single cluster and $C(r)$ reaches 1 for any spatial length r. It is interesting to investigate the time correlation function in the complex phase synchronization region, e.g., the region of phase IV, where the direction phase series are irregular because of the fact that as $\mu>\mu_0$ the logistic map becomes chaotic. Figure 6 shows the dependence of the time correlation function $C(\Delta t)$ on the time step Δt in the case of $\mu=3.8$ for a single site. For each coupling strength ϵ, the values of $C(\Delta t)$ oscillate between 1 and -1. When $\epsilon=0$, the amplitude is small. As the coupling becomes stronger, the amplitude of the time correlation function increases obviously. We can see that the correlativity of the direction phase in two-dimensional coupled map lattices is largely affected by the coupling strength both in the simple phase synchronization region and in the complex one.

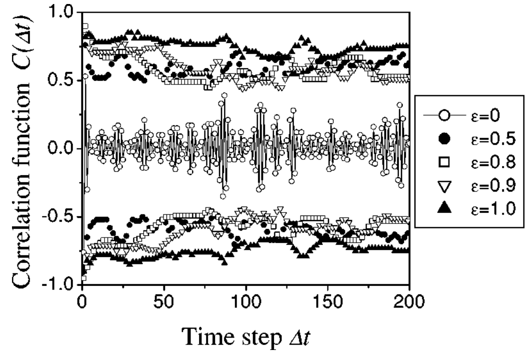

FIG. 6. Relationship between the time correlation function and the time step in the case of $\mu=3.8$ for different coupling strengths $\epsilon=0,0.5,0.8,0.9,1.0$.

V. SUMMARY

In summary, we have studied the behaviors of phase synchronization in a two-dimensional coupled logistic map lattice. By defining a direction phase as the direction of two sequential iterations of the logistic map, we find four different novel kinds of phase synchronization, which correspond to four different regions in a phase diagram. The phase clusters are composed of the lattice sites with the same direction phase at a certain time. We study the features of the phase cluster with simple phase synchronization. In the region of partial phase synchronization, we find that as the coupling strength increases to a critical threshold ϵ_c, there is an infinite cluster in the network. As the coupling further increases to a critical value ϵ_0, the system reaches full phase synchronization. In this case, each site in the lattice has the same direction phase and the phase cluster covers the whole lattice. From the scaling between the percolation probability ρ and $\epsilon-\epsilon_c$, we find that the percolationlike transition of the phase clusters belongs to a different universal class from the ordinary geometrical percolation. We have also studied the spatial and time correlation functions of the phase synchronization, and we find that the correlativity of the phase synchronization is largely affected by the coupling due to the nearest-neighbor interactions.

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